Reversible entanglement beyond quantum operations

Yu-Ao Chen,[*] Xin Wang,[†] Lei Zhang,[‡] and Chenghong Zhu[§]
Thrust of Artificial Intelligence, Information Hub,
The Hong Kong University of Science and Technology (Guangzhou), Guangdong 511453, China

(Received 1 November 2024; accepted 10 February 2025; published 21 March 2025)

We introduce a reversible theory of exact entanglement manipulation by establishing a necessary and sufficient condition for state transfer under trace-preserving transformations that completely preserve the positivity of partial transpose (PPT). Under these free transformations, we show that logarithmic negativity emerges as the pivotal entanglement measure for determining entangled states' transformations, analogous to the role of entropy in the second law of thermodynamics. Previous results proved that entanglement is irreversible under quantum operations that completely preserve PPT and under all nonentangling transformations. In this work, we find that going beyond the complete positivity constraint imposed by standard quantum mechanics enables a reversible theory of exact entanglement manipulation, which may suggest a potential incompatibility between the reversibility of entanglement and the fundamental principles of quantum mechanics.

DOI: 10.1103/PhysRevResearch.7.013297

I. INTRODUCTION

Reversibility is a fundamental concept in many areas of physics, including thermodynamics and quantum mechanics. The second law of thermodynamics governs the direction of heat transfer and the efficiency of energy conversion. With the existence of heat reservoirs, the second law allows for a reversible exchange of work and heat, as exemplified in the Carnot cycle [1]. Based on axiomatic approaches and idealized conditions, it was shown that entropy is the unique function that determines all transformations between comparable equilibrium states [2,3].

Within the realm of quantum information science, reversibility is crucial because it allows for the efficient manipulation of quantum resources. If a process is reversible, it implies that no quantum resources are irretrievably lost during transformations. Through the development of quantum information processing, quantum entanglement has been recognized as an essential resource for various applications, including quantum communication [4], quantum computation [5,6], quantum sensing [7], and cryptography [8]. Understanding the reversibility of entanglement is thus pivotal for quantum information and fuels the debate surrounding the axiomatization of entanglement theory. This discourse is largely driven by the parallels drawn with thermodynamics, which fosters the potential proposition of a single entanglement measure, analogously to entropy, that could potentially govern all entanglement transformations. Such progress in understanding entanglement reversibility would not only mirror thermodynamic properties, but also contribute to the axiomatization of entanglement theory.

Reversibility of entanglement pertains to the process of asymptotic entanglement manipulation. For pure quantum states, this process is reversible, meaning that entanglement can be manipulated and then restored to its original state [9] through local operations and classical communication (LOCC). However, this asymptotic entanglement reversibility does not apply to mixed states [10–14], meaning that once entanglement is manipulated, it cannot be restored to its initial state under LOCC. The irreversibility of quantum entanglement under LOCC presents a stark contrast to the principles of thermodynamics [15], where certain processes are inherently reversible. This irreversibility also underscores the impossibility of developing a single measure [16] capable of governing all entanglement transformations under LOCC, suggesting that a deeper understanding of entanglement manipulation is essential for possible reversibility.

That reversible entanglement theory could exist remains a fundamental problem in quantum information theory [17]. In the pursuit of a reversible entanglement theory, broader classes of operations beyond LOCC can be considered to potentially reduce the gap between entanglement cost and distillable entanglement. However, this approach has yet to yield success as Wang and Duan [18] proved that entanglement was irreversible under quantum operations that completely maintain the positivity of partial transpose (PPT), a meaningful set of quantum operations that includes all LOCC operations. This result [18] also implies the irreversibility in the resource theory of entanglement with nonpositive partial transpose (NPT). Furthermore, Lami and Regula [19] showed that entanglement theory was irreversible under all

[*]Contact author: yuaochen@hkust-gz.edu.cn
[†]Contact author: felixxinwang@hkust-gz.edu.cn
[‡]Contact author: lei.zhang@connect.hkust-gz.edu.cn
[§]Contact author: czhu854@connect.hkust-gz.edu.cn

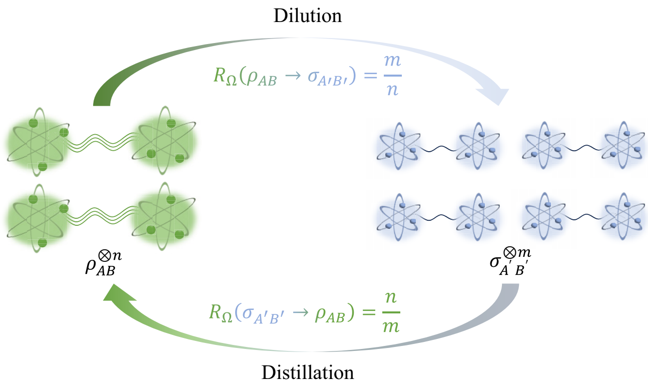


FIG. 1. Exactly reversible interconvertion between quantum states ρ_{AB} and $\sigma_{A'B'}$. Consider n copies of $\rho_{AB}^{\otimes n}$ and m copies of $\sigma_{A'B'}^{\otimes m}$, the forward direction indicates the exact conversion from $\rho_{AB}^{\otimes n}$ to $\sigma_{A'B'}^{\otimes m}$ with rate $R_\Omega(\rho_{AB}^{\otimes n} \to \sigma_{A'B'}^{\otimes m}) = m/n$, while the backward direction shows the exact conversion from $\sigma_{A'B'}^{\otimes m}$ to $\rho_{AB}^{\otimes n}$ with rate $R_\Omega(\sigma_{A'B'}^{\otimes m} \to \rho_{AB}^{\otimes n}) = n/m$. Two states can be exactly interconverted reversibly if $R_\Omega(\rho_{AB}^{\otimes n} \to \sigma_{A'B'}^{\otimes m})R_\Omega(\sigma_{A'B'}^{\otimes m} \to \rho_{AB}^{\otimes n}) = 1$.

nonentangling (NE) transformations, which were positive maps that did not produce entanglement. Recent remarkable works by Hayashi and Yamasaki [20] and Lami [21] showed the proofs of the generalized quantum Stein's lemma [22,23], which led to the reversibility of all quantum resource theories under asymptotically resource nongenerating operations [24]. This prompts the question of whether the identified operations were the only ones capable of ensuring resource reversibility, or if alternative sets of operations could also achieve a single-letter resource measure.

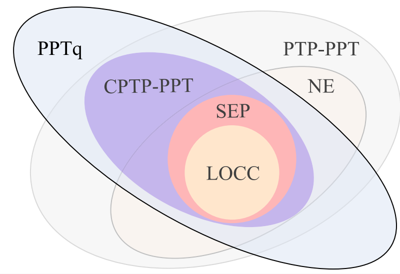
In this paper, we introduce a reversible theory of exact entanglement manipulation (as presented in Fig. 1), showing a possible counterpart of entanglement manipulation to the second law of thermodynamics. This reversible theory operates under transformations that completely preserve the positivity of partial transpose, which are called PPT quasioperations (PPTq operations) throughout the paper. Our key result is that logarithmic negativity fully determines entangled transformations under PPTq operations, i.e.,

$$\rho \xrightarrow{\text{PPTq}} \sigma \iff E_N(\rho) \geqslant E_N(\sigma), \tag{1}$$

which means logarithmic negativity plays an analogous role of entropy in the second laws of thermodynamics. Based on this necessary and sufficient condition of state transformation (cf. Theorem 1), we prove that the logarithmic negativity determines exact distillable entanglement and exact entanglement cost. We further show the reversibility of exact entanglement manipulation under PPTq operations (cf. Theorem 3), showing that

$$R_{\text{PPTq}}(\rho_{AB} \to \sigma_{A'B'}) = \frac{E_N(\rho_{AB})}{E_N(\sigma_{A'B'})}, \tag{2}$$

where $R_{\text{PPTq}}(\rho_{AB} \to \sigma_{A'B'})$ is the asymptotic ratio for exact state transformation under PPTq operations. We further establish an inequality chain of the entanglement manipulation rates for PPTq operations (cf. Theorem 4), presenting a distinction between this reversible theory of entanglement beyond quantum operations and standard entanglement theory.



Class of Operations	Abbreviation
Local Operations and Classical Communication	LOCC
Separable Channels	SEP
Completely PPT-preserving Channels	CPTP-PPT
Non-entangling Operations	NE
Positive and Trace-preserving and PPT-preserving Maps	PTP-PPT
Hermitian-preserving and Trace-preserving PPT Maps	PPTq

FIG. 2. Schematic hierarchy of operations. The pictured inclusions among LOCC, SEP, CPTP-PPT, NE, PTP-PPT, and PPTq. The main focus of this paper is entanglement manipulation under PPTq operations.

While our research establishes a reversible theory of exact entanglement manipulation, it is crucial to note that the allowed transformations in this theory are beyond the boundaries of quantum mechanics. Indeed, generating no entanglement during some state transformations necessitates a reduction of physical law as evidenced in Sec. III. The above reversibility of exact entanglement manipulation under PPT quasioperations may suggest that the coexistence of entanglement reversibility and quantum mechanics might be mutually incompatible.

II. REVERSIBILITY OF EXACT ENTANGLEMENT MANIPULATION

In this section, our focus is centered in exploring the possibility of reversible entanglement manipulation. To thoroughly characterize the fundamental limits of entanglement transformation, we briefly venture beyond the confines of the quantum realm in the context of this section, focusing on entanglement transformations beyond quantum operations. Following the idea of axiomatic approaches, we consider transformations that demand the weakest possible requirements for entanglement manipulation, relaxing the manipulating objects from quantum states to quasistates [cf. Eq. (B2)].

We specifically introduce the PPT quasioperations (PPTq operations), which are Hermitian-preserving and trace-preserving maps that completely preserve the positivity of partial transpose. The set of PPTq operations is shorthanded as PPTq. The underlying intuition is that PPT states can only have bound entanglement [25], which are useless in entanglement distillation. Furthermore, as depicted in Fig. 2, PPTq operations encompass all operations in LOCC, SEP, and CPTP-PPT, whereas some nonentangling and PTP-PPT operations do not fall within the scope of PPTq.

Definition 1. An HPTP bipartite map $\mathcal{N}_{AB\to A'B'}$ is called a PPT quasioperation (PPTq operation) if $\mathrm{T}_{B'} \circ \mathcal{N}_{AB\to A'B'} \circ \mathrm{T}_B$ is completely positive.

In the above definition, T_B denotes the partial transpose operation, as defined in Eq. (B1). All detailed notations are provided in Appendix B. Next, we are going to first establish the necessary and sufficient condition for perfect transformations between quasistates. It turns out surprising that the logarithmic negativity [26,27] is the key to fully characterize the transformations under PPTq operations. We also note that the following theorem directly applies to the more restricted case for transformations between quantum states.

Theorem 1. For two bipartite states or quasi-states ρ and σ, there exists $\mathcal{N}_{AB\to A'B'} \in$ PPTq such that $\mathcal{N}(\rho) = \sigma$ if and only if

$$E_N(\rho) \geqslant E_N(\sigma), \tag{3}$$

where $E_N(\rho) = \log_2 \|\rho^{T_B}\|_1$ is the logarithmic negativity.

Proof ($\Rightarrow$). Construct a linear map $\mathcal{M} = \mathrm{T}_{B'} \circ \mathcal{N} \circ \mathrm{T}_B$. Since $\mathcal{N}_{AB\to A'B'} \in$ PPTq, we know that $\mathcal{M}$ is completely positive and trace-preserving. By construction, it holds that $\mathcal{M}(\rho^{T_B}) = \sigma^{T_{B'}}$. Since any CPTP operation does not increase the trace norm, we immediately have that $\|\rho^{T_B}\|_1 \geqslant \|\sigma^{T_{B'}}\|_1$ and hence $E_N(\rho) \geqslant E_N(\sigma)$ by the monotonicity of the $\log_2$ function.

($\Leftarrow$) The key idea to show this direction is to construct a CPTP map $\mathcal{M}$ such that $\mathcal{M}(\rho^{T_B}) = \sigma^{T_{B'}}$ based on the assumption that $\|\rho^{T_B}\|_1 \geqslant \|\sigma^{T_{B'}}\|_1$, which could guarantee that $\mathcal{N} = \mathrm{T}_{B'} \circ \mathcal{M} \circ \mathrm{T}_B$ is a HPTP and PPT map that can successfully transform ρ to σ. Consider the spectral decomposition $\rho^{T_B} = \sum_j r_j |j\rangle\langle j| = R_+ - R_-$ with $R_+ = \sum_{j:r_j\geqslant 0} r_j |j\rangle\langle j|$ and $R_- = -\sum_{j:r_j<0} r_j |j\rangle\langle j|$. Here we denote the projections for positive and negative parts as

$$P_+ = \sum_{j:r_j\geqslant 0} |j\rangle\langle j|, \quad P_- = \sum_{j:r_j<0} |j\rangle\langle j|, \quad P_+ + P_- = I. \tag{4}$$

Without loss of generality, we assume that $E_N(\sigma) > 0$. Let us then denote $\sigma^{T_{B'}} = \sum_n s_n |n\rangle\langle n|$ and choose some $k \in \mathbb{N}$ such that $s_k \geqslant 0$. We further could write $\sigma^{T_{B'}} = \widetilde{S}_+ - S_- + s_k |k\rangle\langle k|$ with

$$\widetilde{S}_+ = \sum_{n:s_n\geqslant 0,\, n\neq k} s_n |n\rangle\langle n|, \quad S_- = -\sum_{n:s_n<0} s_n |n\rangle\langle n|. \tag{5}$$

We construct the following CPTP map:

$$\mathcal{M}(\omega) = \frac{\mathrm{Tr}(P_+\omega)}{\mathrm{Tr}\, R_+}\widetilde{S}_+ + \frac{\mathrm{Tr}(P_-\omega)}{\mathrm{Tr}\, R_-} S_- + f(\omega)|k\rangle\langle k|, \tag{6}$$

with $f(\omega) = \mathrm{Tr}\,\omega - \mathrm{Tr}(P_+\omega)\,\mathrm{Tr}\,\widetilde{S}_+/\mathrm{Tr}\, R_+ - \mathrm{Tr}(P_-\omega)\,\mathrm{Tr}\, S_-/\mathrm{Tr}\, R_-$. This construction guarantees the condition of TP as the trace of the right-hand side (R.H.S.) of Eq. (6) is equal to $\mathrm{Tr}\,\omega$. To see that this map is also CP, we only need to show that $f(\omega) \geqslant 0$. Note that the prerequisite $E_N(\rho) \geqslant E_N(\sigma)$ implies that $\frac{\mathrm{Tr}\,\widetilde{S}_+}{\mathrm{Tr}\, R_+} \leqslant 1$ and $\frac{\mathrm{Tr}\, S_-}{\mathrm{Tr}\, R_-} \leqslant 1$, thus we have that

$$f(\omega) \geqslant \mathrm{Tr}\,\omega - \mathrm{Tr}(P_+\omega) - \mathrm{Tr}(P_-\omega) = 0. \tag{7}$$

Notably, $\mathcal{M}$ in Eq. (6) is a measure-and-prepare channel [28]. We then apply this channel $\mathcal{M}$ to ρ^{T_B} and obtain

$$\mathcal{M}(\rho^{T_B}) = \mathcal{M}(R_+ - R_-) = \widetilde{S}_+ - S_- + s_k|k\rangle\langle k| = \sigma^{T_{B'}}. \tag{8}$$

■

This result implies that logarithmic negativity emerges as the pivotal entanglement measure for determining the transformations between entangled states, which is analogous to the role of entropy in the second law of thermodynamics.

Reversibility of exact entanglement manipulation under PPTq operations

Entanglement distillation and entanglement dilution are two vital operational tasks in entanglement manipulation. Entanglement distillation [29–37] involves the transformation of a large number of identically and independently distributed (i.i.d.) copies of a certain state into as many ebits as possible. Conversely, entanglement dilution [38–43] is concerned with the reverse process, turning ebits into as many copies of the original state. These procedures are typically carried out by means of LOCC.

Here, we are going to solve the rates for both exact distillable entanglement and exact entanglement cost under PPTq operations, which are shown to be equal to the logarithmic negativity of the state.

Theorem 2. For any bipartite state or quasi-state ρ_{AB}, it holds that

$$E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho_{AB}) = E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\rho_{AB}) = E_N(\rho_{AB}), \tag{9}$$

where $E^{\mathrm{exact}}_{D,\mathrm{PPTq}}$ and $E^{\mathrm{exact}}_{C,\mathrm{PPTq}}$ are the exact distillable entanglement and exact entanglement cost under PPTq operations, formally defined in Appendix C.

Proof. The proof of this theorem is divided into two parts. We are going to first show $E_N(\rho_{AB}) \leqslant E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho_{AB})$ and then show $E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\rho_{AB}) \leqslant E_N(\rho_{AB})$.

Let us consider an n copy of the state ρ_{AB} and try to transform it to as many ebits as possible. By Theorem 1, we know that there exists one $\Lambda \in$ PPTq and maximally entangled state $\Phi^d_{A'B'}$ with $d = \lfloor 2^{nE_N(\rho_{AB})} \rfloor$ such that

$$\Lambda\big(\rho_{AB}^{\otimes n}\big) = \Phi^d_{A'B'}, \tag{10}$$

since $E_N(\rho_{AB}^{\otimes n}) = nE_N(\rho_{AB}) \geqslant \log_2 \lfloor 2^{nE_N(\rho_{AB})} \rfloor = \log_2 d = E_N(\Phi^d_{A'B'})$. Thus, by the definition of exact entanglement distillation, we have

$$E^{(1)}_{0,D,\mathrm{PPTq}}\big(\rho_{AB}^{\otimes n}\big) \geqslant \log_2 \lfloor 2^{nE_N(\rho_{AB})} \rfloor, \tag{11}$$

which leads to

$$E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho_{AB})g = \lim_{n\to\infty} \frac{1}{n} E^{(1)}_{0,D,\mathrm{PPTq}}\big(\rho_{AB}^{\otimes n}\big) \tag{12}$$

$$\geqslant \lim_{n\to\infty} \frac{1}{n} \log_2 \lfloor 2^{nE_N(\rho_{AB})} \rfloor \tag{13}$$

$$= E_N(\rho_{AB}). \tag{14}$$

For the reverse task of entanglement dilution, let us consider n copy of the state ρ_{AB} and try to transform as few ebits as possible to prepare $\rho_{AB}^{\otimes n}$. By Theorem 1, we know that there exists one $\Lambda \in$ PPTq and maximally entangled state $\Phi^d_{A'B'}$ with $d = \lceil 2^{nE_N(\rho_{AB})} \rceil$ such that

$$\Lambda\big(\Phi^d_{A'B'}\big) = \rho_{AB}^{\otimes n}, \tag{15}$$

since $E_N(\rho_{AB}^{\otimes n}) = nE_N(\rho_{AB}) \leqslant \log_2 \lceil 2^{nE_N(\rho_{AB})} \rceil = \log_2 d = E_N(\Phi^d_{A'B'})$. Therefore, by the definition of exact entanglement cost, we have

$$E^{(1)}_{0,C,\mathrm{PPTq}}\big(\rho_{AB}^{\otimes n}\big) \leqslant \log_2\lceil 2^{nE_N(\rho_{AB})}\rceil, \tag{16}$$

which leads to

$$E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\rho_{AB}) = \lim_{n\to\infty}\frac{1}{n}E^{(1)}_{0,C,\mathrm{PPTq}}\big(\rho_{AB}^{\otimes n}\big) \tag{17}$$

$$\leqslant \lim_{n\to\infty}\frac{1}{n}\log_2\lceil 2^{nE_N(\rho_{AB})}\rceil \tag{18}$$

$$= E_N(\rho_{AB}). \tag{19}$$

Note that exact transformations between ρ and Φ^2 guarantees the inequality

$$E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho) \leqslant E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\rho). \tag{20}$$

Thus, combining Eqs. (20), (12), and (17), we arrive at

$$E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho_{AB}) = E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\rho_{AB}) = E_N(\rho_{AB}). \tag{21}$$

■

The above result already implies a collapse of two important entanglement measures. As the logarithmic negativity quantifies both exact entanglement cost and exact distillable entanglement, we want to note that it also explains why logarithmic negativity has been fruitfully used in the theory of quantum entanglement. For example, previous works showed that the logarithmic negativity is an upper bound to distillable entanglement [26] and possesses an operational interpretation as the exact entanglement cost under CPTP-PPT operations for certain classes of quantum states [39,44].

Furthermore, using results on exact entanglement cost and exact distillable entanglement, we are now able to compute the reversibility of asymptotic exact entanglement manipulation under PPTq operations, where the definition of this quantity is given in Appendix C. Note that the reversibility here does not consider the positivity of manipulations, i.e., the practical difficulty of implementing the state manipulations, which is further discussed in Sec. III.

Theorem 3. For any two bipartite states ρ_{AB} and $\sigma_{A'B'}$, the asymptotic exact entanglement transformation rate is given by

$$R_{\mathrm{PPTq}}(\rho_{AB}\to\sigma_{A'B'}) = \frac{E_N(\rho_{AB})}{E_N(\sigma_{A'B'})}, \tag{22}$$

which implies the reversibility of asymptotic exact entanglement manipulation under PPTq operations, i.e.,

$$R_{\mathrm{PPTq}}(\rho_{AB}\to\sigma_{A'B'}) \times R_{\mathrm{PPTq}}(\sigma_{A'B'}\to\rho_{AB}) = 1. \tag{23}$$

Proof. The key idea of this proof is to use the maximally entangled state for intermediate exchange between states, that is, $R_{\mathrm{PPTq}}(\rho_{AB}\to\Phi^2_{\hat{A}\hat{B}}) = E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho_{AB}) = E_N(\rho_{AB})$ and $R_{\mathrm{PPTq}}(\Phi^2_{\hat{A}\hat{B}}\to\sigma_{A'B'}) = E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\sigma_{A'B'})^{-1} = E_N(\sigma_{A'B'})^{-1}$. Therefore, we could obtain the transformation ratio as

$$R_{\mathrm{PPTq}}(\rho_{AB}\to\sigma_{A'B'}) \tag{24}$$

$$\geqslant R_{\mathrm{PPTq}}\big(\rho_{AB}\to\Phi^2_{\hat{A}\hat{B}}\big) \times R_{\mathrm{PPTq}}\big(\Phi^2_{\hat{A}\hat{B}}\to\sigma_{A'B'}\big) \tag{25}$$

$$= E_N(\rho_{AB}) \times E_N(\sigma_{A'B'})^{-1}. \tag{26}$$

This inequality is actually an equality. To see this, assume $R_{\mathrm{PPTq}}(\rho_{AB}\to\sigma_{A'B'}) > E_N(\rho_{AB}) \times E_N(\sigma_{A'B'})^{-1}$. Then

$$R_{\mathrm{PPTq}}(\rho_{AB}\to\sigma_{A'B'}) \times R_{\mathrm{PPTq}}(\sigma_{A'B'}\to\rho_{AB}) \tag{27}$$

$$> \frac{E_N(\rho_{AB})}{E_N(\sigma_{A'B'})^{-1}} \times \frac{E_N(\sigma_{A'B'})}{E_N(\rho_{AB})^{-1}} > 1, \tag{28}$$

which contradicts with the physical definition of R_{PPTq} since transforming ρ back to itself shall not generate extra resources. Such contradiction implies

$$R_{\mathrm{PPTq}}(\rho_{AB}\to\sigma_{A'B'}) = E_N(\rho_{AB}) \times E_N(\sigma_{A'B'})^{-1}, \tag{29}$$

and hence the rest of statements follow. ■

As we establish the tight connection for exact measures $E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\rho)$ and $E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho)$ in Theorem 2, we finally arrive at the inequality chain of the entanglement manipulation rates for PPTq operations. We also take in consideration the tempered negativity E_N^τ [19] in the chain, as shown in the following theorem.

Theorem 4. For any bipartite state ρ, it holds that

$$E_N^\tau(\rho) \leqslant E_{C,\mathrm{PPTq}}(\rho) \leqslant E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\rho) \tag{30}$$

$$= E_N(\rho) = E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho) = E_{D,\mathrm{PPTq}}(\rho). \tag{31}$$

Sketch of Proof. Note that $E_{C,\mathrm{PPTq}}(\rho) \leqslant E^{\mathrm{exact}}_{C,\mathrm{PPTq}}(\rho)$ and $E^{\mathrm{exact}}_{D,\mathrm{PPTq}}(\rho) \leqslant E_{D,\mathrm{PPTq}}(\rho)$ follow by Eqs. (C6) and (C3), respectively. Since Theorem 2 bridges the gap between these two exact measures, it is sufficient to prove the two endpoints in the inequality chain. A full proof can be found in Appendix D.

Remark 1. In the standard quantum resource theory, the achievable rate of entanglement dilution typically surpasses that of entanglement distillation. However, the introduction of PPTq operations reverses this kind of operational inequality. The uncanny phenomenon is attributed to the unique property of quasioperations, which can be decomposed into positive and negative operation components. Such property allows PPTq operations to "borrow" additional entanglement resources from seemingly out of nowhere. Specifically, these components can generate states with extra distillable entanglement. Once the desired transformations are accomplished, this borrowed resource can then be effectively "returned" by combining the positive and negative components to form the target state. This unique feature allows for the achievable rate of entanglement distillation to surpass that of entanglement dilution, presenting a surprisingly reversal of the usual operational inequality in standard resource theory.

Notably, even though the proof demonstrates $E_{C,\mathrm{PPTq}}(\rho) \leqslant E_{D,\mathrm{PPTq}}(\rho)$, the absence of asymptotic continuity in the logarithmic negativity E_N suggests that $E_{C,\mathrm{PPTq}}(\rho)$ does not necessarily equal $E_N(\rho)$, which is the same with other existing entanglement theories under quantum operations.

III. PHYSICAL IMPLEMENTABILITY OF TRANSFORMATIONS UNDER PPTq OPERATIONS

While keeping the efficacy and power in exact reversible entanglement, the set of PPTq operations contains nonphysical operations (e.g., maps that are PPT but not CP). To deepen our understanding of the complexities associated with the manipulation of quantum entanglement, this section is dedicated to an analytical exploration of the physical implementability of such processes within the given setting. We introduce a quantity that characterizes the degree to which the laws of physics must be depreciated to enable entanglement non-generating transformations. By "entanglement nongenerating," we mean that any physical operation used to realize such transformation cannot generate entanglement resource, i.e., needs to preserve the set of separable states. Building on this context, we present two inspiring examples that reveals how logarithm negativity could affect the physical implementability of exact state transformations under PPTq operations.

In the realm of quantum operations, a general HPTP operation $\mathcal{N}$ may not be directly realizable within a quantum system. To circumvent this limitation, one prevalent strategy is to employ a quasiprobability decomposition in the same spirit of [45–48]. This technique entails decomposing $\mathcal{N}$ into a linear combination of two physical operations $\mathcal{N}_1, \mathcal{N}_2$, expressed as $\mathcal{N} = c\mathcal{N}_1 - (c-1)\mathcal{N}_2$ with $c \geqslant 1$ representing a suitably chosen coefficient. The merit of this decomposition lies in its ability to implement $\mathcal{N}$ statistically, as the expectation value under the operation $\mathcal{N}$ can be decomposed as $\mathrm{Tr}[\mathcal{N}(\rho)\cdot O] = c\,\mathrm{Tr}[\mathcal{N}_1(\rho)\cdot O] - (c-1)\,\mathrm{Tr}[\mathcal{N}_2(\rho)\cdot O]$. By physically sampling the operations $\mathcal{N}_1$ and $\mathcal{N}_2$ with probabilities $c/(2c-1)$ and $(c-1)/(2c-1)$, respectively, one can estimate $\mathrm{Tr}[\mathcal{N}(\rho)\cdot O]$ through classical postprocessing, and the complexity of overall sampling times is $\mathcal{O}[(2c-1)^2]$. That is, $\mathcal{N}$ would become more expensive to simulate when $2c-1$ increases.

When $\mathcal{N}$ is entanglement nongenerating, both $\mathcal{N}_1, \mathcal{N}_2$ are required to preserve the set of separable states. Here, to focus on the minimal hardness of physically implementing $\mathcal{N}$, we relax such requirement to be completely PPT-preserving. Then one can accordingly define the physical implementability of transformations between two states as following.

Definition 2. From ρ to σ, the physical implementability of entanglement nongenerating state transformations under PPTq operations is defined as

$$\begin{aligned}\nu(\rho\to\sigma) := \log_2 \min\{2c-1\ |\mathcal{N}(\rho)=\sigma,\\ \mathcal{N} = c\mathcal{N}_1-(c-1)\mathcal{N}_2,\\ \mathcal{N}\in \mathrm{PPTq},\\ \mathcal{N}_{1,2}\in \text{CPTP-PPT}\}.\end{aligned} \quad (32)$$

$\nu(\rho\to\sigma)$ is a practical indicator for assessing the physical feasibility of entanglement reversibility between these two states. In instances where ν is zero, ρ can be transformed to σ via physical PPTq operations. Conversely, when ν is greater than zero, the reversibility between ρ and σ must be constructed using nonphysical operations. The degree of difficulty to achieve such reversibility would get larger as ν increases. In the case where ν approaches infinity, the reversibility would not exist unless additional entanglement resource is introduced to the system.

The physical implementability can be computed via the following semidefinite programming (SDP) [49–51] that evaluates to $2^{\nu(\rho\to\sigma)}$:

$$\min_{\mathcal{J}_{\mathcal{N}_{1,2}},c} \quad 2c-1, \quad (33a)$$

$$\text{s.t.} \quad \mathrm{Tr}_{AB}[(\rho^T\otimes I)\mathcal{J}_{\mathcal{N}}] = \sigma, \quad (33b)$$

$$\mathrm{Tr}_2 J_{\mathcal{N}_1} = cI, \quad \mathrm{Tr}_2 J_{\mathcal{N}_2} = (c-1)I, \quad (33c)$$

$$\mathcal{J}_{\mathcal{N}}^{T_{B'B}}, \mathcal{J}_{\mathcal{N}_j}^{T_{B'B}} \succeq 0\ \forall\, j, \quad (33d)$$

$$\mathcal{J}_{\mathcal{N}} = \mathcal{J}_{\mathcal{N}_1} - \mathcal{J}_{\mathcal{N}_1}, \quad \mathcal{J}_{\mathcal{N}_j}\succeq 0\ \forall\, j, \quad c\geqslant 0. \quad (33e)$$

We also consider the SDP that evaluates the physical implementability of ε-error entanglement manipulation under PPTq operations, and the details are deferred to in Appendix E. To show the nonphysical nature of transformations under PPTq operations, we select the initial state $\rho=\rho_F$ as an isotropic state [52,53] of a two-qutrit system

$$\rho_F = (1-F)(I-\Phi^3)/8 + F\Phi^3, \quad (34)$$

where $\Phi^3 = 1/3\sum_{i,j=0}^{2}|ii\rangle\langle jj|$ is the normalized maximally entangled state. On the other hand, we choose the target state σ as an antisymmetric state [54]

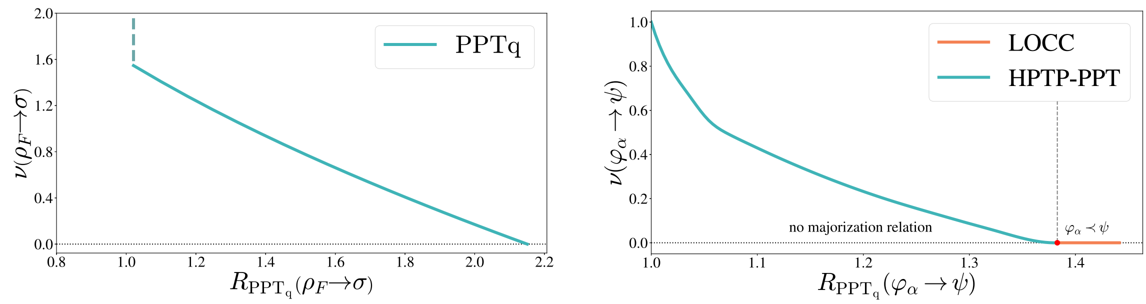
$$\sigma = \tfrac{1}{6}(I - \mathrm{SWAP}_3), \quad (35)$$

where $\mathrm{SWAP}_3 = \sum_{i,j=0}^{2}|ij\rangle\langle ji|$ is the nine-dimensional SWAP gate. As illustrated in Fig. 3(a), before the asymptotic exact transformation rate $R_{\mathrm{PPTq}}(\rho_F\to\sigma)$ reaches the maximum, it is evident that the physical implementability from an initial state ρ_F to a target state σ is nonzero. The experiment results suggest that transformations between some states necessitate a relaxation on laws of physics to maintain reversibility, even as the logarithmic negativity of input state is greater than that of output state.

Relation with LOCC transformations of pure states

According to Theorem 1, the existence of exact state transformations under PPTq operations between two bipartite states ρ, σ is fully determined by the inequality of their logarithmic negativity. Notably, when these states are pure, this domination also applies to exact state transformations under LOCC operations [55], with the feasibility of such transformation is governed by a relation called "majorization." This notable parallel prompts us to investigate the relationship between logarithmic negativity and majorization within the context of pure states. In this section, using the physical implementability, we show that majorization can imply the inequality of logarithmic negativity, while the converse may not hold in general.

Suppose φ, ψ are pure quantum states. In the context of quantum state majorization, we say that a state H is majorized by another state K (denoted as $H \prec K$) if and only if H can be decomposed into a convex combination of $U_jKU_j^\dagger$ for a set of unitary matrices U_j. Then [55] states that a transformation from state φ to state ψ via LOCC is feasible if and only if $\varphi_A \prec \psi_A$. Combining Theorem 1, one can have the known



(a) Transformations from ρ_F to σ, for $F \in (0.1, 1]$.

(b) Transformations from φ_α to ψ, for $\alpha \in (0.1, 1]$.

FIG. 3. The figures demonstrating the nonphysical nature of exact state transformations under PPTq operations between two-qutrit states. (a) depicts the attempted transformation from an isotropic state ρ_F to an antisymmetric state σ, with the blue dashed line emphasizing the transformation's infeasibility. (b) The transformation between two bipartite pure states, with the input state being parameterized by a coefficient α. The red dot in this figure marks the juncture at which the transformation shifts from a nonphysical to a physical regime.

result

$$\varphi_A \prec \psi_A \Rightarrow E_N(\varphi) \geqslant E_N(\psi). \tag{36}$$

However, the converse does not generally hold, as there are instances where states can only be transformed using nonphysical completely PPT-preserving operations. This can be illustrated by considering the two-qutrit pure states

$$|\varphi_\alpha\rangle \propto \frac{1}{5}|00\rangle + \frac{1}{10}|11\rangle + \alpha|22\rangle, \tag{37}$$

$$|\psi\rangle = \frac{1}{\sqrt{5}}|00\rangle + \frac{2}{\sqrt{5}}|11\rangle, \tag{38}$$

for a real coefficient α. As depicted in Fig. 3(b), when the possible transformation turns to be physical by increasing the conversion ratio to around 1.37, the LOCC transformation from $\varphi_\alpha = |\varphi_\alpha\rangle\langle\varphi_\alpha|$ to $\psi = |\psi\rangle\langle\psi|$ becomes attainable. Such an abrupt shift suggests that without additional constraints, the logarithmic negativity cannot directly influence the majorization relation between pure states. It also emphasizes that extra conversion cost is necessary to maintain reversibility of exact state transformations.

IV. DISCUSSIONS

Our work demonstrated the reversibility of exact entanglement transformations under PPTq operations. This reversibility established a parallel between entanglement manipulation and the second law of thermodynamics, particularly when operating under idealized conditions. The logarithmic negativity was the key entanglement measure in this reversible entanglement theory to determine the exact transformation between entangled states, serving a role analogous to entropy in the realm of thermodynamics. The advent of a reversible theory of exact entanglement manipulation under PPTq operations paved the way for further exploration into the smallest subset of quantum operations or maps nestled within the set of PPTq operations that could guarantee the reversibility of asymptotic entanglement manipulation. Our work also opens a possible avenue to study quantum entanglement beyond free quantum operations and the physical implementability for quantum entanglement transformations.

The reversibility of exact entanglement manipulation, along with the irreversibility brought by the positivity of nonentangling transformations [19], may suggest a potential incompatibility between the foundational principles of quantum mechanics and the reversibility of quantum entanglement. In addition, as a recent work [56] showed that the reversibility of quantum resources could happen when relaxing to probabilistic transformations, it will also be interesting to study the interplay between reversibility, success probability, and positivity of allowed transformations of quantum resources.

ACKNOWLEDGMENTS

The authors are listed in alphabetical order. We would like to thank Bartosz Regula and Chengkai Zhu for helpful discussions and thank Benchi Zhao, Xuanqiang Zhao, and Mingrui Jing for their useful comments. This work was partially supported by the National Key R&D Program of China (Grant No. 2024YFE0102500); the National Natural Science Foundation of China (Grant. No. 12447107); the Guangdong Provincial Quantum Science Strategic Initiative (Grants No. GDZX2403008 and No. GDZX2403001); the Guangdong Provincial Key Lab of Integrated Communication, Sensing, and Computation for Ubiquitous Internet of Things (Grant No. 2023B1212010007); the Quantum Science Center of Guangdong-Hong Kong-Macao Greater Bay Area; and the Education Bureau of Guangzhou Municipality.

APPENDIX A: TABLE OF SOME RESOURCE THEORIES

In Table I, we summarize the mainstream resource theories, while our work establishes a reversible framework for exact entanglement manipulation under PPTq operations.

APPENDIX B: SYMBOLS AND NOTATIONS

1. Notations for bipartite systems

In this work, we use the symbols A and B to denote the finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ associated with Alice and Bob systems, respectively. We denote the dimension of $\mathcal{H}_A$ and $\mathcal{H}_B$ as d_A and d_B. The identity operator on system A is denoted as $\mathbb{1}_A$. A quantum state on system A is a positive-semi-definite operator ρ_A with trace one. The trace norm of ρ is denoted as $\|\rho\|_1 = \mathrm{Tr}(\sqrt{\rho^\dagger\rho})$. Let $\{|i\rangle\}_{i=0}^{d-1}$ be a standard computational basis, then a standard maximally entangled state of Schmidt rank d is $\Phi^d = 1/d\sum_{i,j=0}^{d-1}|ii\rangle\langle jj|$. A Bell

TABLE I. Mainstream resource theories. This work presents a reversible theory of exact entanglement manipulation under PPTq operations, where asymptotic exact entanglement transformation is reversible.

Paradigm	Class of operations	Free resource	Resource	Reversible?
Thermodynamics	–	Heat	Work	√
Coherence [57]	Incoherent	Incoherent states	Coherent states	✗
Coherence [58]	Maximally incoherent	Incoherent states	Coherent states	√
Entanglement [10]	LOCC	Separable states	Entangled states	✗
NPT entanglement [18]	CPTP-PPT	PPT states	NPT states	✗
Entanglement [19]	Nonentangling	Separable states	Entangled states	✗
NPT Entanglement (This Work)	PPTq	Quasistates w. zero E_N	Quasistates w. positive E_N	√

state Φ^2 is alternatively called an ebit. The partial transpose of a bipartite quantum state ρ_{AB} on system B is denoted by $\rho_{AB}^{T_B}$, defined as

$$\mathrm{T}_B(\rho_{AB}) = \sum_{i,i'=0}^{d_B-1} (\mathbb{1}_A \otimes |i\rangle\langle i'|_B)\rho_{AB}(\mathbb{1}_A \otimes |i\rangle\langle i'|_B), \qquad \text{(B1)}$$

where T_B is the corresponding partial transpose map. Further, ρ_{AB} is said to be a PPT state, if it admits positive partial transpose, i.e., $\rho_{AB}^{T_B}$ is positive-semi-definite. Throughout the paper, we take the logarithm to be base two unless stated otherwise.

2. Properties of linear maps

Let $\mathcal{N}$ be a linear map. $\mathcal{N}$ is Hermitian-preserving (HP) if it maps any Hermitian operator to another Hermitian operator; $\mathcal{N}$ is trace-preserving (TP) if it preserves the traces of input operators; $\mathcal{N}$ is HPTP if $\mathcal{N}$ is HP and TP. $\mathcal{N}$ is positive if it maps any positive-semi-definite operator to another positive-semi-definite operator, and is called completely positive (CP) if this positivity is preserved on any extended reference system.

3. Properties of bipartite linear maps

Let $\mathcal{N}$ be a bipartite linear map. $\mathcal{N}$ is a local operation and classical communication (LOCC) if it is a composition of a (finite) sequence of quantum instruments. $\mathcal{N}$ is a separable map if it can be written as the sum of bipartite product maps, and the class of separable channels is denoted as SEP. $\mathcal{N}$ is positivity-of-the-partial-transpose-preserving (PPT-preserving) if it preserves the set of PPT states [59,60], and is considered completely PPT-preserving (PPT) if this property holds on any extended reference system. The set of completely PPT-preserving channels is shorthanded as CPTP-PPT. Combining the physical conditions, positive and trace-preserving maps that preserve the set of separable state are nonentangling (NE) operations; the set of positive, trace-preserving, and PPT-preserving maps is denoted as PTP-PPT. We note that free operations beyond LOCC are of importance to advance our understanding of quantum entanglement (see, e.g., [24,29,30,61–63]).

4. Quasistates

A quasistate is a mathematical entity that encapsulates the intricacies of a probabilistic quantum system. Despite the nonphysicality in quantum mechanics, the class of quasistates is the largest possible set that can be represented by physical quantum states under a statistical meaning. Formally, the definition of this class is given as

$$\widetilde{\mathcal{S}} = \left\{ \sum_j c_j\rho_j : \rho_j \in \mathcal{S}, c_j \in \mathbb{R} \text{ s.t. } \sum_j c_j = 1 \right\}, \qquad \text{(B2)}$$

where $\mathcal{S}$ is the set of quantum states. Note that $\widetilde{\mathcal{S}}$ is the set of all Hermitian matrices with trace one.

APPENDIX C: ENTANGLEMENT MANIPULATION

1. Quasistate transformations

Throughout this paper, we allow both state and quasistate transformations, viewing them as operator transformations under linear maps. Specifically, we say a bipartite quasistate ρ can be transformed into another bipartite quasistate σ under Ω operations, if there exists a bipartite linear map $\mathcal{N} \in \Omega$ such that $\mathcal{N}(\rho) = \sigma$. This broader framework of quasistate transformation enables us to define distillable entanglement and entanglement cost also for bipartite quasistates, similar to how we would for standard quantum states. This approach offers a fresh perspective on understanding the limits of manipulating quantum entanglement.

2. Entanglement distillation

The maximally entangled state plays a role as the currency in quantum information since it has become a key ingredient in many quantum information processing tasks (e.g., teleportation [64], superdense coding [65], and quantum cryptography [8]). It is important to understand how many maximally entangled states we can obtain from a source of less entangled states using free operations. Imagine that Alice and Bob share a large supply of identically prepared states, and they want to convert these states to high-fidelity Bell pairs. Let Ω represent a set of free operations or allowed transformations. The one-shot zero-error, or exact distillable entanglement, under Ω operations of quantum state or quasistate ρ_{AB} is defined as $E_{0,D,\Omega}^{(1)}(\rho_{AB}) = \sup_{\Lambda\in\Omega}\{\log_2 d : \Phi_{\hat{A}\hat{B}}^d = \Lambda_{AB\to\hat{A}\hat{B}}(\rho_{AB})\}$. The zero error, or exact distillable entanglement, of a bipartite state or quasistate state ρ_{AB}, under Ω operations is defined as

$$E_{D,\Omega}^{\text{exact}}(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} E_{0,D,\Omega}^{(1)}(\rho_{AB}^{\otimes n}). \qquad \text{(C1)}$$

For entanglement distillation with asymptotically vanishing error, the rate is quantified via distillable entanglement. The distillable entanglement of a bipartite state or quasistate ρ_{AB}, under the Ω operations, is defined as

$$E_{D,\Omega}(\rho_{AB}) = \sup\left\{r : \lim_{n\to\infty}\left[\inf_{\Lambda\in\Omega}\left\|\Lambda\left(\rho_{AB}^{\otimes n}\right) - \Phi_{\hat{A}\hat{B}}^{2^{rn}}\right\|_1\right] = 0\right\}. \tag{C2}$$

As distillable entanglement quantifies the fundamental limit of entanglement distillation and related task of quantum communication, substantial efforts have been made to obtain its accurate estimation and fundamental properties [29–37]. As exact entanglement distillation is more restricted, it holds that

$$E_{D,\Omega}^{\text{exact}}(\rho) \leqslant E_{D,\Omega}(\rho). \tag{C3}$$

3. Entanglement dilution

The reverse task of entanglement distillation is called entanglement dilution. At this time, Alice and Bob share a large supply of Bell pairs and they aim to convert rn Bell pairs to n high fidelity copies of the desired state $\rho^{\otimes n}$ using suitable free operations. Let Ω represent a set of free operations, which, for example, can be LOCC or CPTP-PPT. The one-shot zero-error, or exact entanglement cost, of a bipartite state or quasistate ρ_{AB} under the Ω operations is defined as $E_{0,C,\Omega}^{(1)}(\rho_{AB}) = \inf_{\Lambda\in\Omega}\{\log_2 d : \rho_{AB} = \Lambda_{\hat{A}\hat{B}\to AB}(\Phi_{\hat{A}\hat{B}}^d)\}$. The zero error, or exact entanglement cost, of ρ_{AB} under the Ω operations is defined as

$$E_{C,\Omega}^{\text{exact}}(\rho_{AB}) = \lim_{n\to\infty}\frac{1}{n}E_{0,C,\Omega}^{(1)}\left(\rho_{AB}^{\otimes n}\right). \tag{C4}$$

Previous works [38–43] showed progress towards understanding the exact entanglement cost of quantum states.

For the case of asymptotically vanishing error of entanglement dilution, the rate is quantified via entanglement cost. The concise definition of entanglement cost using Ω operations is given as follows:

$$E_{C,\Omega}(\rho_{AB}) = \inf\left\{r : \lim_{n\to\infty}\inf_{\Lambda\in\Omega}\left\|\rho_{AB}^{\otimes n} - \Lambda\left(\left(\Phi_{\hat{A}\hat{B}}^2\right)^{\otimes rn}\right)\right\|_1 = 0\right\}. \tag{C5}$$

When LOCC is free, entanglement cost is given by the regularization of the entanglement of formation [66], which is shown to be nonadditive [67]. Further efforts have been made to improve understanding of the entanglement cost in specific and general quantum states [18,54,68,69]. As exact entanglement dilution is more restricted, it holds that

$$E_{C,\Omega}(\rho) \leqslant E_{C,\Omega}^{\text{exact}}(\rho). \tag{C6}$$

4. Exact entanglement transformations

The ratio of state conversion plays an integral role in the manipulation of quantum resources such as entanglement. We summarize the mainstream theories for reversible state conversions in Table I. In this paper, we mainly focus on the asymptotic exact state conversion. It describes the process of converting one state into another exactly as the number of copies approaches infinity under certain free operations. Let Ω represent a set of free operations. The asymptotic conversion ratio of exact entanglement transformation from a state or quasistate ρ_{AB} to another state or quasistate $\sigma_{\hat{A}\hat{B}}$ is defined as

$$R_\Omega(\rho_{AB}\to\sigma_{A'B'}) = \sup\left\{\frac{m}{n} : \exists n_0 : \forall n \geqslant n_0, \exists\Lambda_n\in\Omega : \Lambda_n\left(\rho_{AB}^{\otimes n}\right) = \sigma_{A'B'}^{\otimes m}\right\}. \tag{C7}$$

Two states can be exactly interconverted reversibly if $R_\Omega(\rho_{AB}\to\sigma_{A'B'})\times R_\Omega(\sigma_{A'B'}\to\rho_{AB}) = 1$.

APPENDIX D: INEQUALITY CHAIN FOR ENTANGLEMENT MEASURES UNDER PPTq

In this section, we rigorously demonstrate how the inequality chain in Theorem 4

$$\begin{aligned} E_N^\tau(\rho) &\leqslant E_{C,\text{PPTq}}(\rho) \leqslant E_{C,\text{PPTq}}^{\text{exact}}(\rho) \\ &= E_N(\rho) = E_{D,\text{PPTq}}^{\text{exact}}(\rho) = E_{D,\text{PPTq}}(\rho) \end{aligned} \tag{D1}$$

is built. We start from the construction of two endpoints in the chain, i.e., prove $E_N^\tau(\rho) \leqslant E_{C,\text{PPTq}}(\rho_{AB})$ (Proposition D 1) and $E_{D,\text{PPTq}}(\rho_{AB}) \leqslant E_N(\rho_{AB})$ (Proposition D 2). For the temper logarithm negativity part, we first need to extend the definition of tempered negativity in [19] to fit in the regime of quasistates.

Definition D 1. Let σ_{AB}, ρ_{AB} be two quasistates. The tempered negativity between σ_{AB} and ρ_{AB} is defined as

$$\begin{aligned} N_\tau(\sigma_{AB} \mid \rho_{AB}) = \sup\{&\text{Tr}[X\sigma_{AB}] : \|X^{T_B}\|_\infty \leqslant 1, \\ &\|X\|_\infty = \text{Tr}[X\rho_{AB}]\}. \end{aligned} \tag{D2}$$

Further, the tempered negativity of ρ_{AB} is denoted by

$$N_\tau(\rho_{AB}) = N_\tau(\rho_{AB} \mid \rho_{AB}). \tag{D3}$$

Here, we restate three properties of N_τ, and show that the original proof in [19] remains true after the extension. Then we are ready to give the proposition of inequality.

Lemma D 1. For any quasistate ω_{AB} and state ρ_{AB},

(a) $\|\sigma_{AB}^{T_B}\|_1 \geqslant N_\tau(\sigma_{AB} \mid \rho_{AB})$,

(b) $\|\sigma_{AB} - \rho_{AB}\|_1 \leqslant \varepsilon \Rightarrow N_\tau(\sigma_{AB}|\rho_{AB}) \geqslant (1-\varepsilon)N_\tau(\rho_{AB})$, and

(c) $N_\tau(\rho_{AB}^{\otimes n}) \geqslant N_\tau(\rho_{AB})^n$.

Proof. (**a**) This property follows by the fact that $\|\sigma_{AB}\|_1 = \sup\{\text{Tr}[X\sigma_{AB}] : \|X^{T_B}\|_\infty \leqslant 1\}$.

(**b**) Note that the Hölder's inequality holds for arbitrary complex matrices. Hence, this properties still holds by Eq. (S46) in [19].

(**c**) Since ρ_{AB} is a quantum state, this property holds from the same reasoning in [19]. ■

Proposition D1. For any bipartite state ρ_{AB}, it holds that

$$E_{C,\mathrm{PPTq}}(\rho_{AB}) \geqslant E_N^\tau(\rho). \tag{D4}$$

Proof. This proof mainly follows from Lemma D 1 and the idea of chain inequalities in [19]. For $r = E_{C,\mathrm{PPTq}}(\rho_{AB})$, there exists a sequence of PPT quasioperations $\{\Lambda_n\}_n$ such that

$$\sigma_{n,\hat{A}\hat{B}} = \Lambda_n\big(\Phi_{\hat{A}\hat{B}}^{2\lfloor rn\rfloor}\big), \quad \varepsilon_n = \big\|\sigma_{n,\hat{A}\hat{B}} - \rho_{AB}^{\otimes n}\big\|_1 \quad \text{and} \lim_{n\to\infty}\varepsilon_n = 0. \tag{D5}$$

Then for all n, Theorem 1 implies $E_N(\Phi_{\hat{A}\hat{B}}^{2\lfloor rn\rfloor}) \geqslant E_N(\sigma_{n,\hat{A}\hat{B}})$ and hence

$$\begin{aligned}2^{\lfloor rn\rfloor} &\geqslant \big\|(\sigma_{n,\hat{A}\hat{B}})^{T_B}\big\|_1\\ &\overset{\text{(a)}}{\geqslant} N_\tau\big(\sigma_{n,\hat{A}\hat{B}}\big|\rho_{AB}^{\otimes n}\big) \overset{\text{(b)}}{\geqslant} (1-\varepsilon_n)N_\tau\big(\rho_{AB}^{\otimes n}\big) \overset{\text{(c)}}{\geqslant} (1-\varepsilon_n)N_\tau(\rho_{AB})^n.\end{aligned} \tag{D6}$$

■

Taking $n\to\infty$ on both sides gives $r = \lim_{n\to\infty}\lfloor rn\rfloor/n \geqslant E_N^\tau(\rho_{AB})$.

For the entanglement distillation part, the proposition can be given as follows.

Proposition D2. For any state or quasistate ρ_{AB}, it holds that

$$E_{D,\mathrm{PPTq}}(\rho_{AB}) \leqslant E_N(\rho_{AB}). \tag{D7}$$

Proof. This proof mainly follows the proof from [26]. For $r = E_{D,\mathrm{PPTq}}(\rho_{AB})$, there exists a sequence of PPT quasioperations $\{\Lambda_n\}_n$ such that

$$\sigma_{n,\hat{A}\hat{B}} = \Lambda_n\big(\rho_{AB}^{\otimes n}\big), \quad \varepsilon_n = \big\|\sigma_{n,\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^{2\lfloor rn\rfloor}\big\|_1 \quad \text{and} \lim_{n\to\infty}\varepsilon_n = 0. \tag{D8}$$

Then for all n, Theorem 1 implies $E_N(\rho_{AB}^{\otimes n}) \geqslant E_N(\sigma_{n,\hat{A}\hat{B}})$ and hence

$$\begin{aligned}\big\|\big(\rho_{AB}^{\otimes n}\big)^{T_B}\big\|_1 &\geqslant \|(\sigma_{n,\hat{A}\hat{B}})^{T_B}\|_1\\ &= \big\|\big(\Phi_{\hat{A}\hat{B}}^{2\lfloor rn\rfloor}\big)^{T_B} + \big(\sigma_{n,\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^{2\lfloor rn\rfloor}\big)^{T_B}\big\|_1 &(\text{D9})\\ &\geqslant \big\|\big(\Phi_{\hat{A}\hat{B}}^{2\lfloor rn\rfloor}\big)^{T_B}\big\|_1 - \big\|\big(\sigma_{n,\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^{2\lfloor rn\rfloor}\big)^{T_B}\big\|_1 &(\text{D10})\\ &\geqslant (1-\varepsilon_n)2^{\lfloor rn\rfloor}, &(\text{D11})\end{aligned}$$

where the last inequality follows from the fact $\|X^{T_B}\|_1 \leqslant d\|X\|_1$ for $X \in \mathrm{Herm}(\mathcal{H}_d)$. Taking $n\to\infty$ on both sides, the additivity of logarithm negativity gives

$$\begin{aligned}E_N(\rho_{AB}) &\geqslant \lim_{n\to\infty}\frac{1}{n}\log_2(1-\varepsilon_n)2^{\lfloor rn\rfloor}\\ &= \lim_{n\to\infty}\frac{\lfloor rn\rfloor}{n} + \frac{\log_2(1-\varepsilon_n)}{n} = r.\end{aligned} \tag{D12}$$

■

We are ready to construct the whole chain.

Theorem D1. For any bipartite state ρ, it holds that

$$\begin{aligned}&E_N^\tau(\rho) \leqslant E_{C,\mathrm{PPTq}}(\rho) \leqslant E_{C,\mathrm{PPTq}}^{\mathrm{exact}}(\rho)\\ &= E_N(\rho) = E_{D,\mathrm{PPTq}}^{\mathrm{exact}}(\rho) = E_{D,\mathrm{PPTq}}(\rho).\end{aligned} \tag{D13}$$

Proof. For all bipartite state ρ,

$$\begin{aligned}E_N^\tau(\rho) &\leqslant E_{C,\mathrm{PPTq}}(\rho) &&(\text{Proposition D 1}), &(\text{D14})\\ &\leqslant E_{C,\mathrm{PPTq}}^{\mathrm{exact}}(\rho) &&[\text{Eq. (C6)}] &(\text{D15})\\ &= E_N(\rho) = E_{D,\mathrm{PPTq}}^{\mathrm{exact}}(\rho) &&(\text{Theorem 2}) &(\text{D16})\\ &\leqslant E_{D,\mathrm{PPTq}}(\rho) &&[\text{Eq. (C3)}] &(\text{D17})\\ &\leqslant E_N(\rho). &&(\text{Proposition D 2}) &(\text{D18})\end{aligned}$$

■

APPENDIX E: SDP OF ENTANGLEMENT MANIPULATION WITH ERROR UNDER PPTq

Definition E1. A (finite) collection of pairs of real coefficients and physical operations $\{(c_j, \mathcal{N}_j)\}_j$ is said to be a quasiprobability decomposition of a linear map $\mathcal{N}$ if $\mathcal{N} = \sum_j c_j\mathcal{N}_j$.

One may notice that the idea of quasiprobability decomposition in Sec. III only involves two pairs. Indeed, if $\mathcal{N}$ has a quasiprobability decomposition since $\mathcal{N}$ is linear and properties discussing in this paper is linear, one can assume that such decomposition is composed by two pairs, i.e., $\mathcal{N} = c_1\mathcal{N}_1 - c_2\mathcal{N}_2$ for some positive real numbers c_1, c_2. Further, when $\mathcal{N}$ is trace-preserving, $c_1 + c_2 = 1$.

Definition E2 (Entanglement nongenerating). Let $\mathcal{N}$ be a linear map acting on bipartite states. We say that $\mathcal{N}$ is entanglement nongenerating if $\mathcal{N}$ preserves the set of separable states, and there exists a quasiprobability decomposition $\{(c_j, \mathcal{N}_j)\}_j$ of this map such that each component $\mathcal{N}_j$ preserves the set of separable states. Moreover, we say that $\mathcal{N}$ is an entanglement nongenerating transformation under PPTq operations if $\mathcal{N} \in$ PPTq.

Definition E3 (ε-error physicalness of the map). For any two bipartite state ρ and σ, the physical implementability of entanglement nongenerating state transformations under PPTq operations with error ε is defined as

$$\nu^\varepsilon(\rho\to\sigma) := \log_2\min\{2c-1 \mid \|\mathcal{N}(\rho)-\sigma\|_\infty < \varepsilon,\quad \mathcal{N} = c\mathcal{N}_1 - (c-1)\mathcal{N}_2 \in \mathrm{PPTq},\quad \mathcal{N}_{1,2} \in \text{CPTP-PPT}\}. \tag{E1}$$

The following SDP computes the quantity $2^{\nu^\varepsilon(\rho\to\sigma)}$,

$$\min_{\mathcal{J}_{\mathcal{N}_1},\mathcal{J}_{\mathcal{N}_2},c} \quad 2c-1, \tag{E2a}$$

$$\text{s.t.} \quad -\varepsilon I \preceq \mathrm{Tr}_{AB}[(\rho^T \otimes I)\mathcal{J}_{\mathcal{N}}] - \sigma \preceq \varepsilon \quad (\varepsilon\text{-error transformation}), \tag{E2b}$$

$$\mathrm{Tr}_2 J_{\mathcal{N}_1} = cI, \quad \mathrm{Tr}_2 J_{\mathcal{N}_2} = (c-1)I \quad (\text{trace-preserving}), \tag{E2c}$$

$$\mathcal{J}_{\mathcal{N}}^{T_{B'B}}, \mathcal{J}_{\mathcal{N}_j}^{T_{B'B}} \succeq 0 \ \forall\, j \quad (\text{completely PPT-preserving}), \tag{E2d}$$

$$\mathcal{J}_{\mathcal{N}} = \mathcal{J}_{\mathcal{N}_1} - \mathcal{J}_{\mathcal{N}_2}, \quad \mathcal{J}_{\mathcal{N}_j} \succeq 0 \ \forall\, j, c \geqslant 0 \quad (\text{Hermitian-preserving}). \tag{E2e}$$

[1] S. Carnot, *Réflexions sur la Puissance Motrice du Feu* (Vrin, Paris, 1979), Vol. 26.
[2] E. H. Lieb and J. Yngvason, The physics and mathematics of the second law of thermodynamics, Phys. Rep. **310**, 1 (1999).
[3] R. Giles, *Mathematical Foundations of Thermodynamics: International Series of Monographs on Pure and Applied Mathematics* (Elsevier, Amsterdam, 2016), Vol. 53.
[4] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Entanglement-assisted classical capacity of noisy quantum channels, Phys. Rev. Lett. **83**, 3081 (1999).
[5] D. Bruß and C. Macchiavello, Multipartite entanglement in quantum algorithms, Phys. Rev. A **83**, 052313 (2011).
[6] R. Jozsa and N. Linden, On the role of entanglement in quantum-computational speed-up, Proc. R. Soc. Lond. A **459**, 2011 (2003).
[7] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, Rev. Mod. Phys. **89**, 035002 (2017).
[8] A. K. Ekert, Quantum cryptography based on Bell's theorem, Phys. Rev. Lett. **67**, 661 (1991).
[9] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A **53**, 2046 (1996).
[10] G. Vidal and J. I. Cirac, Irreversibility in asymptotic manipulations of entanglement, Phys. Rev. Lett. **86**, 5803 (2001).
[11] G. Vidal, W. Dür, and J. I. Cirac, Entanglement cost of bipartite mixed states, Phys. Rev. Lett. **89**, 027901 (2002).
[12] K. G. H. Vollbrecht, R. F. Werner, and M. M. Wolf, Irreversibility of entanglement distillation for a class of symmetric states, Phys. Rev. A **69**, 062304 (2004).
[13] M. F. Cornelio, M. C. de Oliveira, and F. F. Fanchini, Entanglement irreversibility from quantum discord and quantum deficit, Phys. Rev. Lett. **107**, 020502 (2011).
[14] D. Yang, M. Horodecki, R. Horodecki, and B. Synak-Radtke, Irreversibility for all bound entangled states, Phys. Rev. Lett. **95**, 190501 (2005).
[15] M. Horodecki, J. Oppenheim, and R. Horodecki, Are the laws of entanglement theory thermodynamical? Phys. Rev. Lett. **89**, 240403 (2002).
[16] M. Horodecki, P. Horodecki, and J. Oppenheim, Reversible transformations from pure to mixed states and the unique measure of information, Phys. Rev. A **67**, 062104 (2003).
[17] M. B. Plenio, Problem 20, https://oqp.iqoqi.oeaw.ac.at/reversible-entanglement-manipulation.
[18] X. Wang and R. Duan, Irreversibility of asymptotic entanglement manipulation under quantum operations completely preserving positivity of partial transpose, Phys. Rev. Lett. **119**, 180506 (2017).
[19] L. Lami and B. Regula, No second law of entanglement manipulation after all, Nat. Phys. **19**, 184 (2023).
[20] M. Hayashi and H. Yamasaki, Generalized quantum Stein's lemma and second law of quantum resource theories, arXiv:2408.02722.
[21] L. Lami, A solution of the generalised quantum Stein's lemma, arXiv:2408.06410.
[22] M. Berta, F. G. S. L. Brandão, G. Gour, L. Lami, M. B. Plenio, B. Regula, and M. Tomamichel, On a gap in the proof of the generalised quantum Stein's lemma and its consequences for the reversibility of quantum resources, Quantum **7**, 1103 (2023).
[23] K. Fang, G. Gour, and X. Wang, Towards the ultimate limits of quantum channel discrimination, arXiv:2110.14842.
[24] F. G. S. L. Brandão and M. B. Plenio, A reversible theory of entanglement and its relation to the second law, Commun. Math. Phys. **295**, 829 (2010).
[25] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-state entanglement and distillation: Is there a "bound" entanglement in nature? Phys. Rev. Lett. **80**, 5239 (1998).
[26] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).
[27] M. B. Plenio, Logarithmic negativity: A full entanglement monotone that is not convex, Phys. Rev. Lett. **95**, 090503 (2005).
[28] S. Khatri and M. M. Wilde, Principles of quantum communication theory: A modern approach, arXiv:2011.04672.
[29] E. M. Rains, A semidefinite program for distillable entanglement, IEEE Trans. Inf. Theory **47**, 2921 (2000).
[30] T. Eggeling, K. G. H. Vollbrecht, R. F. Werner, and M. M. Wolf, Distillability via protocols respecting the positivity of partial transpose, Phys. Rev. Lett. **87**, 257902 (2001).
[31] M. Horodecki, P. Horodecki, and R. Horodecki, Asymptotic manipulations of entanglement can exhibit genuine irreversibility, Phys. Rev. Lett. **84**, 4260 (2000).
[32] M. Christandl and A. Winter, Squashed entanglement An additive entanglement measure, J. Math. Phys. **45**, 829 (2004).
[33] X. Wang and R. Duan, Improved semidefinite programming upper bound on distillable entanglement, Phys. Rev. A **94**, 050301(R) (2016).
[34] X. Wang and R. Duan, Nonadditivity of Rains' bound for distillable entanglement, Phys. Rev. A **95**, 062322 (2017).
[35] F. Leditzky, N. Datta, and G. Smith, Useful states and entanglement distillation, IEEE Trans. Inf. Theory **64**, 4689 (2017).

[36] X. Wang, Pursuing the fundamental limits for quantum communication, IEEE Trans. Inf. Theory **67**, 4524 (2021).
[37] C. Zhu, C. Zhu, and X. Wang, Estimate distillable entanglement and quantum capacity by squeezing useless entanglement, IEEE J. Sel. Areas Commun. **42**, 1850 (2024).
[38] B. M. Terhal and P. Horodecki, Schmidt number for density matrices, Phys. Rev. A **61**, 040301(R) (2000).
[39] K. Audenaert, M. B. Plenio, and J. Eisert, Entanglement cost under positive-partial-transpose-preserving operations, Phys. Rev. Lett. **90**, 027901 (2003).
[40] W. Matthews and A. Winter, Pure-state transformations and catalysis under operations that completely preserve positivity of partial transpose, Phys. Rev. A **78**, 012317 (2008).
[41] X. Wang and M. M. Wilde, Cost of Quantum Entanglement Simplified, Phys. Rev. Lett. **125**, 040502 (2020).
[42] X. Wang and M. Wilde, Errata for "Cost of quantum entanglement simplified" and "Exact entanglement cost of quantum states and channels under PPT-preserving operations," Zenodo (2024), doi:10.5281/zenodo.11061607.
[43] L. Lami, F. A. Mele, and B. Regula, Computable entanglement cost under positive partial transpose operations, Phys. Rev. Lett. **134**, 090202 (2025).
[44] S. Ishizaka, Binegativity and geometry of entangled states in two qubits, Phys. Rev. A **69**, 020301(R) (2004).
[45] J. Jiang, K. Wang, and X. Wang, Physical implementability of linear maps and its application in error mitigation, Quantum **5**, 600 (2021).
[46] X. Zhao, L. Zhang, B. Zhao, and X. Wang, Power of quantum measurement in simulating unphysical operations, arXiv:2309.09963.
[47] K. Wang, Z. Song, X. Zhao, Z. Wang, and X. Wang, Detecting and quantifying entanglement on near-term quantum devices, npj Quantum Inf. **8**, 52 (2022).
[48] N. Cao, M. Fitzsimmons, Z. Mann, R. Pereira, and R. Laflamme, Quantum maps between CPTP and HPTP, arXiv:2308.01894.
[49] J. Watrous, *The Theory of Quantum Information*, 1st ed. (Cambridge University Press, Cambridge, England, 2018).
[50] S. Khatri and M. M. Wilde, Principles of quantum communication theory: A modern approach, arXiv:2011.04672.
[51] X. Wang, Semidefinite optimization for quantum information, Ph.D. thesis, University of Technology Sydney, 2018.
[52] M. Horodecki and P. Horodecki, Reduction criterion of separability and limits for a class of protocols of entanglement distillation, arXiv:quant-ph/9708015.
[53] K. Chen, S. Albeverio, and S.-M. Fei, Entanglement of formation of bipartite quantum states, Phys. Rev. Lett. **95**, 210501 (2005).
[54] M. Christandl, N. Schuch, and A. Winter, Entanglement of the antisymmetric state, Commun. Math. Phys. **311**, 397 (2012).
[55] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th ed. (Cambridge University Press, Cambridge, England, 2010).
[56] B. Regula and L. Lami, Reversibility of quantum resources through probabilistic protocols, Nat. Commun. **15**, 3096 (2024).
[57] A. Winter and D. Yang, Operational resource theory of coherence, Phys. Rev. Lett. **116**, 120404 (2016).
[58] A. Streltsov, G. Adesso, and M. B. Plenio, *Colloquium:* Quantum coherence as a resource, Rev. Mod. Phys. **89**, 041003 (2017).
[59] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A **223**, 1 (1996).
[60] A. Peres, Separability criterion for density matrices, Phys. Rev. Lett. **77**, 1413 (1996).
[61] E. M. Rains, Entanglement purification via separable superoperators, Phys. Rev. A **60**, 179 (1999).
[62] E. Chitambar, J. I. de Vicente, M. W. Girard, and G. Gour, Entanglement manipulation beyond local operations and classical communication, J. Math. Phys. **61**, 042201 (2020).
[63] B. Regula, K. Fang, X. Wang, and M. Gu, One-shot entanglement distillation beyond local operations and classical communication, New J. Phys. **21**, 103017 (2019).
[64] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. **70**, 1895 (1993).
[65] C. H. Bennett and S. J. Wiesner, Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, Phys. Rev. Lett. **69**, 2881 (1992).
[66] P. M. Hayden, M. Horodecki, and B. M. Terhal, The asymptotic entanglement cost of preparing a quantum state, J. Phys. A: Math. Gen. **34**, 6891 (2001).
[67] M. B. Hastings, Superadditivity of communication capacity using entangled inputs, Nat. Phys. **5**, 255 (2009).
[68] M. M. Wilde, Entanglement cost and quantum channel simulation, Phys. Rev. A **98**, 042338 (2018).
[69] X. Wang, M. Jing, and C. Zhu, Computable and faithful lower bound for entanglement cost, arXiv:2311.10649.